

# ARTICLES

## SINGLE NOTE DISSONANCE THROUGH HARMONIC SELF-INTERFERENCE

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### ABSTRACT

Musical dissonance is generally understood in terms of two simultaneous notes. However, low frequency notes sound dissonant when played alone on a piano. The explanation proposed in this work is that this dissonance arises from the harmonics of the played note interfering with one another.

Using the piano as a model, perceived dissonance was calculated through the combination of the two-tone dissonance formula with the A-weighting curve and the different harmonic intensities of a piano. Sound spectrums of sample piano notes were used to compare harmonics of low frequency and high frequency notes.

Single note dissonance increased rapidly as note frequency decreased. A-weighting had no qualitative effect on the dissonance-frequency trend, implying a physical and not an aural cause. As verified in the sound spectrums, the lower register note had harmonics closer together, compared to the higher register note. It is thus possible that the harmonics of low notes interfere significantly with each other, thereby producing the observed single-note dissonance.

The simulation produces a score for the perceived dissonance of a single-note played on the piano. This analysis could be adapted in the future to other instruments, including aerophones, as well as integrate timbre, partials, and inharmonics.

La dissonance musicale et pour la plupart compris en matière de deux notes simultanées. Cependant, les notes de basse fréquence semble en désordre quand ils sont joués seules sur un piano. L'explication proposée dans cet œuvre est que cette dissonance se produit des harmoniques des notes qui s'interfèrent.

Utilisant le piano comme exemple, la dissonance perçue a été calculé par la combinaison de la formule de dissonance entre deux notes avec la courbe de pondération A et les intensités harmoniques d'un piano. Les spectres sonores des notes de piano ont été utilisés pour la comparaison des notes de basse et haut fréquences.

La dissonance des seules notes augmentait rapidement alors que les fréquences des notes diminuaient. La pondération A n'avait pas eu un effet sur la tendance entre la dissonance et les fréquences, qui signifie la présence d'une cause physique au lieu d'une cause auditive. Vérifié dans les spectres sonores, la note de registre inférieur avait ses harmoniques plus proches comparés à la note du registre plus haut. C'est alors possible que les harmoniques des notes des registres inférieurs interfèrent considérablement avec les unes les autres, et produit ainsi la dissonance des seules notes observée dans les calculs.

La simulation produit un résultat pour la dissonance perçue sur un piano. Cette analyse pourrait être adaptée dans le futur aux autres instruments, incluant les aérophones, et puis intégrer aussi des analyses de timbre, des tons simples, et des inharmoniques.

### KEY WORDS

Dissonance; Music; Harmonic; Note; Interference

### INTRODUCTION

Music can be defined as the use of sounds, from any number of instruments, to express and evoke feelings and emotions. What makes good music and what makes bad music has been, and will likely remain,

a point of debate and discussion. One intriguing facet of music near the center of that conversation is music's ability to create consonance, as well as its more infamous foil, dissonance. Consonance can be

considered a harmonious or pleasant sound from the combination of musical notes, while dissonance is the unpleasant quality of sound.

Helmutz theorized that dissonance was produced from the interference between two tones to produce displeasing beatings.<sup>1</sup> For strings, this beat theory is applicable by considering interfering standing sine waves on the strings as analogous to their interfering emitted sounds. However, in the field of dissonance, while much has been studied for two notes, more research is possible in terms of the dissonance produced by a single note. Each instrument has a unique timbre, or sound quality. On a piano, as the played note is lowered in pitch, there is a noticeable increase in the roughness, unclarity, or murkiness of the sound quality. This texture is a dissonance caused uniquely from playing a single note.<sup>2</sup> Because this effect is common to many instruments, its explanation must also be adaptable or analogous across instruments.

With inspiration from beat theory, there may be a physical interference in emitted sound waves that is causing this single-note dissonance. The difference from the usual approach is that there are not two notes being played, but rather a single note. An important consideration is to regard the single note not as a single frequency. On traditional instruments, any note played will generate harmonics – frequencies higher than the intended fundamental frequency.<sup>2</sup> It is possible that the notes in the middle and upper registers may be perceived more clearly as pure or single tones, having less audible harmonics. However, the harmonics created from a single note in the lower register may be much more audible to the human ear. These low-register harmonic frequencies, if perceived by the ear, may be close enough to the fundamental frequency and each other to create beats and thus dissonance.

This interference may have implications in the development of new instruments, as well as the tuning of existing instruments. While instruments may create a theoretical fundamental frequency, the sound heard by the human ear may be quite different due to this single-note dissonance effect. Lower-register note self-dissonance is also important to music theory, as well as reconsidering any unnatural use of low pure tones instead of bass notes during acoustic tests.

The mechanism proposed in this work postulates that notes played in the lower registers of instruments can create dissonance when played on their own due to interference from and amongst their own harmonics. A mathematical model of the piano, including harmonics and two-tone dissonance production, is used to analyze whether self-dissonance is significantly greater in lower-register notes than higher-register notes.

## MATERIALS AND METHODS

### *Instrument Model*

The standard 88-note piano is used as a stringed instrument model, played at approximately 60dB.<sup>3</sup> Note frequency increases from left to right on the piano keyboard. The piano utilizes strings bound on both ends. Hence, the corresponding physics can be simulated as a standing wave on the string.

### *Keyboard to frequency formula, 12-TET*

To convert between a note on the piano keyboard with a frequency, the piano was assumed to be tuned using the Twelve-Tone-Equal-Temperament (12-TET) scale. This allows any note  $n$  to be converted into its respective frequency via the formula  $f_n = 2^{\frac{n-49}{12}} \times 440$ .<sup>4</sup>  $f_n$  is the frequency of the  $n^{\text{th}}$  note,  $n$  is the numerical coordinate of the piano keyboard note, and 440 Hz is the frequency of the 49<sup>th</sup> note ( $A_4$ ), held as a standard.

### *Trigonometric assumption of sound waves*

Pure tone sounds are assumed to be held at a constant frequency with constant amplitude. Their periodic nature can be described by a sine wave, with nodes at both ends of the string.

### *Two-tone dissonance formula*

The relative dissonance created from two frequencies is approximated by Jensen in the formula

$$D(g) = I_f I_g (e^{-0.84(\frac{f-f_g}{0.0217+19})} - e^{-1.38(\frac{f-f_g}{0.0217+19})})$$

where  $g$  is the value of the higher frequency note,  $f$  is the value of the lower frequency note, and  $I_g$  and  $I_f$  are their respective intensities.<sup>5,6</sup>

### Harmonic intensities of a piano

The intensities of the harmonics from piano strings vary according to the following values, measured by Helmholtz<sup>7</sup>:

$I_1 = 1.0$ ,  $I_2 = 2.9$ ,  $I_3 = 3.6$ ,  $I_4 = 2.6$ ,  $I_5 = 1.1$ ,  $I_6 = 0.2$ , and  $I_7 = 0.0$ .

### A-weighting curve from equal-loudness contours

To compensate for the human ear's non-uniform perception of the loudness of sounds at different frequencies, the A-weighting curve was used, given by the International Standards Organization as:

$$A(f) = 2.0 + 20 \log_{10} \left( \frac{12200^2 \times f^4}{(f^2 + 12200^2)(f^2 + 20.6^2)(\sqrt{(f^2 + 107.7^2)(f^2 + 737.9^2)})} \right)$$

Where  $A(f)$  is the decibel difference to what the human ear perceives the frequency  $f$  to be, as compared to a standard 1000 Hz sound.<sup>8</sup>

### Computer software

Microsoft® Excel® for Mac 2011 was used for calculations and data plots. MacCRO X was used to obtain sound spectrums from a Yamaha digital piano. Grapher was used for graph generation.

## RESULTS

### 1. Maximum dissonance for two pure tones

The dissonance formula

$$D(g) = I_f I_g (e^{-0.84(\frac{g-f}{0.021f+19})} - e^{-1.38(\frac{g-f}{0.021f+19})})$$

can be rewritten  $D(g) = I_f I_g (e^{-0.84Q} - e^{-1.38Q})$

as by performing a substitution where  $Q = \frac{g-f}{0.021f+19}$ .<sup>6</sup>

The solution for the value of frequency  $g$  ( $g_{\max}$ ) that provides maximum dissonance given frequency  $f$ , via calculus (a full derivation is presented in the Appendix), is  $g_{\max} = 1.01931f + 17.4672$  in Hertz

The two tone dissonance function can be graphed and the maximum dissonance value can be identified (Figure 1).

### 2. Derivation of two-tone perceived dissonance formula

The intensity of a piano being played has been measured at roughly 60dB.<sup>7</sup> Assuming 60dB is registered for the fundamental frequency, the intensities of the harmonics can be calculated through corresponding multiples of the sum of 60dB and the Helmholtz's harmonic intensities. The A-weighting calculation is added to this intensity to compensate for

human perception. The physical restrictions are that (i) there is no dissonance for union ( $f = g$ ), and (ii) an intensity of 0.0 leads to no dissonance.

The resulting perceived dissonance formula:

$$pD(g) = (I_f(60 + A(f)))(I_g(60 + A(f)))(e^{-0.84(\frac{g-f}{0.021f+19})} - e^{-1.38(\frac{g-f}{0.021f+19})})$$

### 3. Mathematical simulation of a single note's harmonics self-interference dissonance

Using the new perceived dissonance formula, this formula can be appropriated to a single note under certain conditions if  $g$  is taken as the harmonic of  $f$ . When using A-weighting, the following data is calculated (Figure 2). Without using A-weighting, the following data is calculated (Figure 3).

### 4. Sound spectrums of sample notes on a piano

As representative of a low register note, a sound spectrum of the  $A_2$  note is sampled (Figure 4). As representative of a high register note, a sound spectrum of the  $A_5$  note is sampled (Figure 5).

## DISCUSSION

This mathematical analysis of the harmonics produced by a piano model is intended to investigate the possible dissonance produced by a single note's own harmonics. The hypothesis is that this harmonic self-interference is the primary factor that leads to the observed dissonant texture of lower register notes, as opposed to the clarity and purity of higher register notes.

There is a relatively strong increase in the perceived dissonance of a single note as its fundamental frequency is decreased towards low-register notes (Figures 2, 3). The qualitative rise in dissonance is similar in both the simulation with A-weighting, as well as without the A-weighting. Therefore, the single-note dissonance is most prevalently a physical phenomenon, and less so due to human aural conditions for perception. As expected, there is relatively very low dissonance in the middle ( $C_4$ ) and upper ( $C_5$  and onward) registers.

The spectrum of the lower register note  $A_2$  (Figure 4) has harmonics that are much closer to one another in terms of absolute frequency, in comparison with the spectrum of the upper register note  $A_5$  (Figure 5). This validates the trigonometric assumption of the bound wave, such that the harmonics of  $A_2$  are closer due to them being multiples of a smaller fundamental

frequency value, in comparison to the higher note  $A_5$ . The use of varying intensities of harmonics increases the accuracy of the simulation. The spectrum's shape in harmonic intensities supports the used harmonic intensity data for the dissonance simulation.

A possible reason for this single-note dissonance phenomenon is due to the closeness of these harmonics. As seen in the spectrums, as the frequency of the note decreases, the closer the harmonics become to one another. This proximity can lead to interference amongst the harmonics themselves, leading to the beats and therefore the observed dissonance. However, in higher register notes these harmonics would be too far apart to have significant interference with one another within the human ear, as calculated in Figure 2 and 3, and observed in Figure 4, compared to Figure 5.

The A-weighting scale is used to calibrate for varying the auditory intensities of the human ear. Without the use of A-weighting the dissonance of lower register notes is increased, likely because the loudness of the sound in total has been increased.

The piano is the simplest model for analysis, due to its bound strings and straightforward construction concept, with a wide frequency range divided visually through its keyboard. While the fundamental loudness may not be exactly 60dB for each note of the piano at all times, the calculations are relative, so this variation can be dismissed for the purposes of this investigation.

A key assumption in this simulation is that all non-fundamental frequencies generated are harmonics; however, this may not be true. Inharmonics occur when overtones are not integer multiples of the fundamental frequency. However, inharmonics are not significant for lower register notes, thereby preserving the validity of the simulation.<sup>9</sup>

Due to the qualitative nature of the sensation of dissonance, it is hard to quantify. However, the two-tone dissonance formula can be validated by solving for  $g_{\max}$ , and comparing it to the experimental result. For the standard  $f = n_{49} = A_4 = 440$  Hz, the formula predicts  $g_{\max} = 465.96$  Hz. This frequency is very close to that of the Minor Second interval of  $n_{50}$ , given as  $n_{50} = A\#_4 = 466.16\text{Hz} \approx 465.96\text{Hz} = g_{\max}$ . The Minor Second interval is considered the most dissonant interval, and has been accurately predicted. Furthermore, the original two-tone dissonance formula by Jensen

is based on the work by Sethares and Benson.<sup>6</sup> The peak dissonance occurs when two frequencies occur at approximately 1/4 of the critical bandwidth.<sup>6</sup> The dissonance formula reflects this peak in dissonance due to the nature of the critical bandwidth, as well as takes into account that the critical bandwidth increases with the registered fundamental frequency. While the specific mechanics of the critical bandwidth is outside the aim and scope of this investigation, the beat theory of Helmholtz is seemingly verified as these beats can be produced within the critical bandwidth and can therefore produce the perceived dissonance.

## CONCLUSION

In agreement with observations, the dissonance produced by a single note increased as its fundamental frequency decreased. By the 4th octave and into subsequent higher octaves, dissonance values were similarly low in comparison with octaves below the 4th. A-weighting had no effect on this qualitative trend.

This investigation was limited to assuming only harmonics are generated in the piano model. A more complete simulation would include partials dependent on the instrument. While the inharmonic effects are not significant in the bass, higher octaves have strong deviation in harmonic frequencies, and may create their own form of single-note dissonance.<sup>9</sup> This analysis can also be adapted to other instruments, including other chordophones or aerophones, to examine the effect of instrument construction. These models are important to form a more complete theory of acoustic dissonance on the mind. The study of dissonance could provide answers and analyses to several questions: how do musical conventions and interpretations of dissonance differ between cultures and genres, and how might this affect the use or scarcity of low-register notes in music.

## ABBREVIATIONS

Hz	Hertz
n	note

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## APPENDIX

*Full derivation of the maximum dissonance for two pure tones*

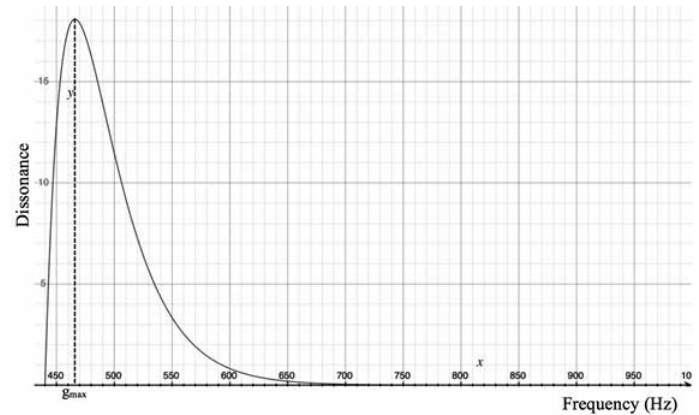
The dissonance formula  $D(g) = I_f I_g (e^{-0.84(\frac{g-f}{0.021f+19})} - e^{-1.38(\frac{g-f}{0.021f+19})})$  can be rewritten as  $D(g) = I_f I_g (e^{-0.84Q} - e^{-1.38Q})$  by performing a substitution where  $Q = \frac{g-f}{0.021f+19}$ .

Solution for the value of frequency g ( $g_{max}$ ) that provides maximum dissonance given frequency f, via calculus:

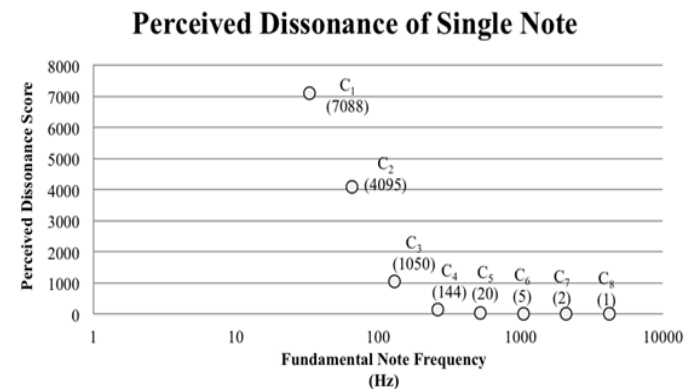
$$\begin{aligned}
 D(g) &= I_f I_g (e^{-0.84Q} - e^{-1.38Q}) \\
 \frac{d(D(g))}{dg} &= 0 = I_f I_g (-0.84(\frac{dQ}{dg})e^{-0.84Q} + 1.38(\frac{dQ}{dg})e^{-1.38Q}) \\
 0 &= -0.84(\frac{dQ}{dg})e^{-0.84Q} + 1.38(\frac{dQ}{dg})e^{-1.38Q} \\
 0.84e^{-0.84Q} &= 1.38e^{-1.38Q} \\
 \frac{0.84}{1.38} &= \frac{e^{-1.38Q}}{e^{-0.84Q}} = e^{-1.38Q+0.84Q} \\
 \ln(\frac{0.84}{1.38}) &= \ln(e^{-0.54Q}) = -0.54Q \\
 \frac{\ln(\frac{0.84}{1.38})}{-0.54} &= Q = \frac{g_{max}-f}{0.021f+19} \\
 g_{max} &= (0.021f+19) \frac{\ln(\frac{0.84}{1.38})}{-0.54} + f
 \end{aligned}$$

For maximum dissonance:  $g_{max} = 1.01931f + 17.4672$  in Hertz

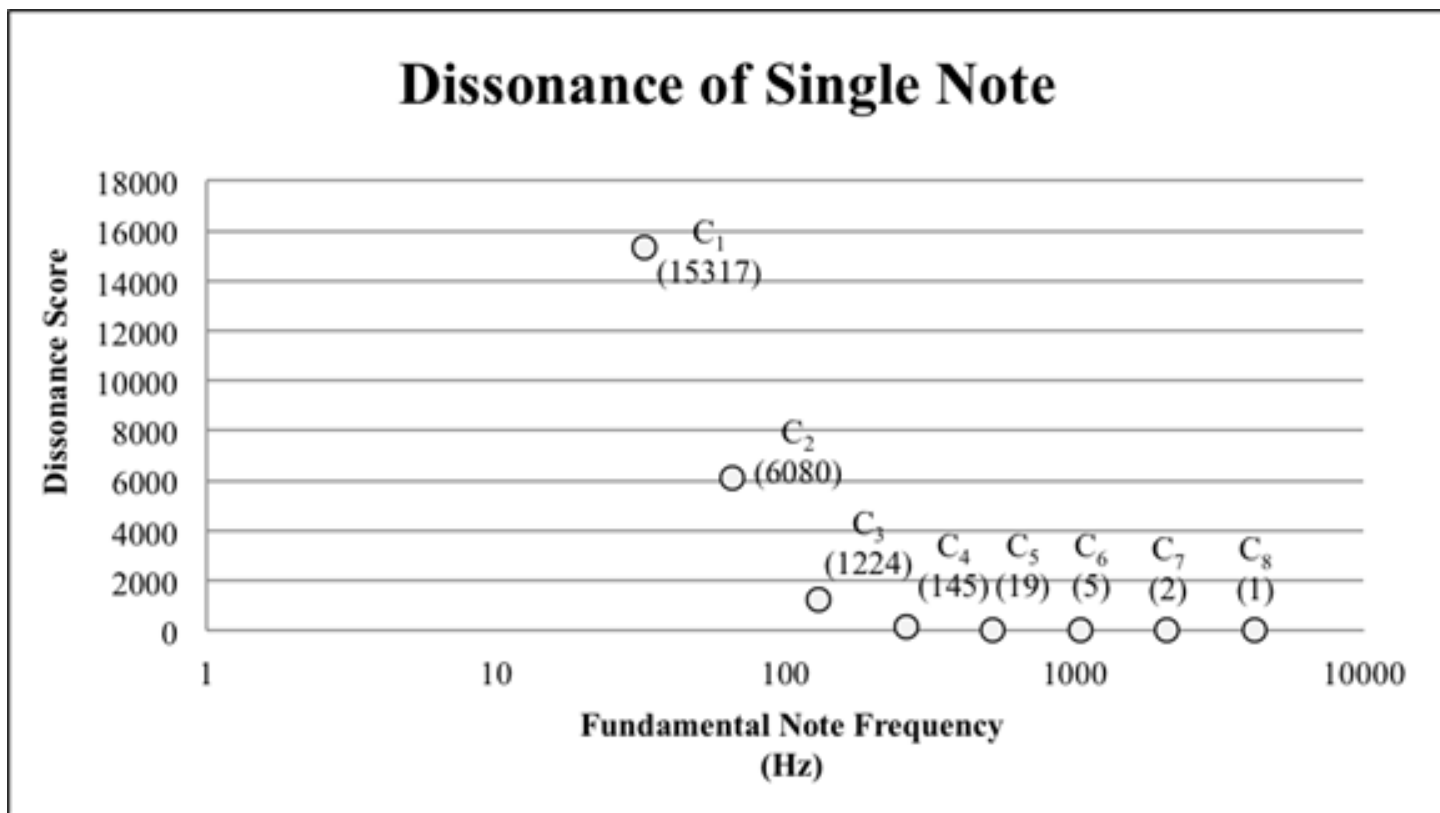
The two tone dissonance function can be graphed and the maximum dissonance value can be identified (Figure 1).



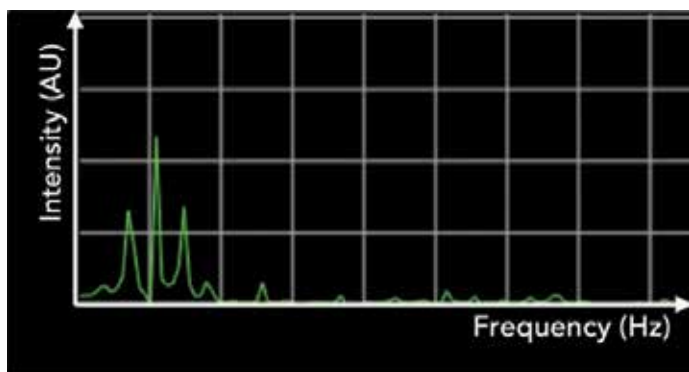
**Figure 1.** Two-tone dissonance curve generated from the two-tone dissonance formula. Frequency f is set to 440 Hz. The dashed line indicates the position of  $g_{max}$ , the frequency value of g for which maximum dissonance is generated.



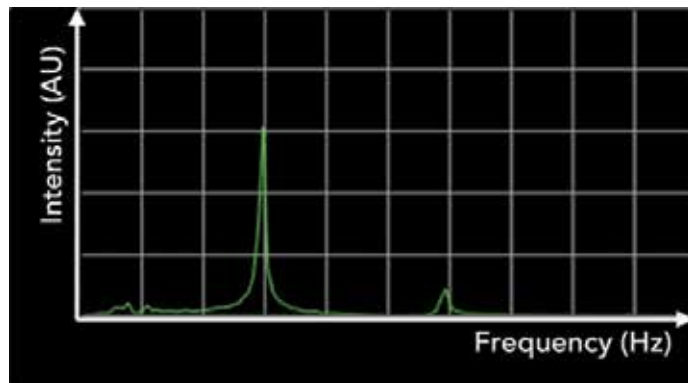
**Figure 2.** Graphical display of the calculated perceived dissonance of a single note played on a piano, using A-weighting. Perceived dissonance values between the fundamental and all successive harmonics, in every combination, are calculated and summed to create the note's final perceived dissonance score. Calculations were performed for all C notes on the 88-note keyboard.



**Figure 3.** Single note dissonance was calculated as in Figure 2; however, the A-weighting was removed (A-weighting parameter set to 0 in all calculations).



**Figure 4.** Image of the sound spectrum of the A2 note on a Yamaha digital piano. The sound wave produced by the piano is deconstructed into separate frequencies (Hz) with the intensity of each frequency component measured (arbitrary units).



**Figure 5.** Image of the sound spectrum of the A5 note on a Yamaha digital piano. The sound wave produced by the piano is deconstructed into separate frequencies (Hz) with the intensity of each frequency component measured (arbitrary units).